

Homework 3 Solution Key

Math 121A, Fall 2025.

Due:

Problem 1. Throughout the following, let $a, b \in \mathbb{R}$ be scalars and let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$ be vectors solving each respective equation. To check whether a given set is a subspace is to check if each vector $a\vec{x} + b\vec{y}$ solves the equation.

(a) This is a subspace, since

$$(ax_1 + by_1) + 2(ax_2 + by_2) + 3(ax_3 + by_3) = a(x_1 + 2x_2 + 3x_3) + b(y_1 + 2y_2 + 3y_3) = 0.$$

(b) This is not a subspace, since e.g. it does not contain the zero vector.

(c) This is not a subspace, since the vectors $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ satisfy the equation $x_1x_2x_3 = 0$, but their sum $\vec{s} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ does not, since $1 \cdot 2 \cdot 1 \neq 0$.

(d) This is a subspace, since

$$(ax_1 + by_1) = ax_1 + by_1 = a(5x_3) + b(5y_3) = 5(ax_3 + by_3).$$

Problem 2. (*MAT 54 review 1*)

(a) The matrix version of this equation is

$$\underbrace{\begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \end{pmatrix}}_M \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}_{\vec{b}}.$$

(b) M “eats” vectors in $\mathbb{R}^4$ and “spits out” vectors in $\mathbb{R}^3$. So, $M : \mathbb{R}^4 \rightarrow \mathbb{R}^3$. Note that M is a 3×4 matrix, since it has 3 rows and 4 columns.

(c) The augmented matrix corresponding to the matrix equation $M\vec{x} = \vec{b}$ is

$$\left(\begin{array}{cccc|c} 0 & -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & -1 & -1 & -1 & 0 \end{array} \right)$$

The RREF of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & 1 & 2/3 \end{array} \right).$$

So, one solution is given by $\vec{x}_p = \begin{pmatrix} 4/3 \\ 2/3 \\ 2/3 \\ 0 \end{pmatrix}$.

Recalling that the range is defined by $\text{ran}(M) = \{\vec{b} \in \mathbb{R}^m : \text{there exists a solution } \vec{x} \in \mathbb{R}^n \text{ to } M\vec{x} = \vec{b}\}$, we see that yes, $\vec{b} \in \text{ran}(M)$.

(d) Looking at RREF, we may compute the null space to be

$$\text{null}(M) = \left\{ \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} \in \mathbb{R}^4 : x = 0, y = 0, \text{ and } w + z = 0 \right\}.$$

Or in other words, $\text{null}(M) = \text{span} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right)$. This means that the solution is not unique, since

$$\dim(\text{null}(M)) > 0.$$

(e) The dimension of S does not change if we shift it, so $\dim(S) = \dim(\text{null})(M) = 1$.

(f) By either looking at RREF or by using the rank-nullity theorem, $\dim \text{range}(M) = 3$.

i. Since the dimension of the output space is 3, this means that $\text{range}(M) = \mathbb{R}^3$, so the linear map is onto. In other words, the equation $M\vec{x} = \vec{c}$ has a solution for any vector $\vec{c} \in \mathbb{R}^3$.

ii. The dimension of S_c is the same if we shift it, so $\dim(S_c) = \dim(S)$.

Problem 3. (MAT 54 review 2)

(a)

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & -3 \end{pmatrix}}_M \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\vec{b}}.$$

(b) Now, $M : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

(c) No. The domain has $\dim(\mathbb{R}^2) = 2$, so by the rank-nullity theorem $\text{rank}(M) \leq 2$, so it cannot span the whole space.

(d) The rref of M is

$$\text{rref}(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\text{rank}(M) = 2$ and $\dim \text{null}(M) = 0$, since there are 2 pivot columns and no free variables.

Problem 4. (Change of coordinates)

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d)

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 5. (*Rotations*)

(a) This matrix is

$$[R_\theta] = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(b) This is exactly the columns of this matrix:

$$[R_\theta] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad [R_\theta] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

(c) The inverse is given by

$$[R_\theta]^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = [R_{-\theta}].$$

In particular, the inverse of a rotation by θ is a rotation by $-\theta$. Geometrically, this just means the inverse of a rotation is rotating the opposite direction by the same angle.

Problem 6. (*Our first quantum state*)

(a) Notice that the first column is a linear multiple of the second column. Thus a basis for $\text{range}(\rho)$ is just the vector $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To find a basis for $\text{null}(\rho)$, we can just pick the orthogonal vector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (check for yourself that $\rho \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$). Both are 1-dimensional spaces.

(b) In this basis, we have

$$[\rho]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(c) Observe that if $\psi = \psi_1 v_1 + \psi_2 v_2$, then its matrix form is $\psi_{\mathcal{B}} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Then, applying ρ ,

$$[\rho]_{\mathcal{B}}[\psi]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$$

In particular, ρ “kills off” the component of ψ in the direction of v_2 , leaving only the “projection of ψ onto v_1 ”.

Problem 7. (*Function spaces*)

(a) We may do this in a variety of ways. Let’s choose 3 x values, say $x = -1, 0, 1$. Then we have

three column vectors corresponding to $\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We may show that these vectors are linearly independent by e.g. showing that the matrix has

$$\det \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2.$$

(b) In this basis, the vector is

$$[p(x)] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}.$$

(c) Since we know the derivative is linear, it only remains to show that it is a map $V \rightarrow V$. Observe that $\frac{\partial}{\partial x}$ must lower the degree of a polynomial: since V consists of polynomials of degree ≤ 2 , the derivative maps V to V . To find its matrix, we check what this operator does to the basis. $\frac{\partial}{\partial x} 1 = 0$, $\frac{\partial}{\partial x} x = 1$, and $\frac{\partial}{\partial x} x^2 = 2x$. Thus its matrix is

$$\left[\frac{\partial}{\partial x}\right] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

To verify this, check that if $p(x) = a + bx + cx^2$, then $\frac{\partial p}{\partial x} = b + 2cx$, so

$$\left[\frac{\partial}{\partial x}\right][p(x)] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix}.$$

(d) No. For instance, it has a nontrivial null space with a basis given by $\{1\}$ (this is just because the derivative of a constant is 0).

- (e) Again, we just need to check what this operator does to a basis. $I(1) = 1$, $I(x) = \frac{1}{2}$, and $I(x^2) = \frac{1}{3}$. Thus,

$$[I] = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

As a sanity check, let $p(x) = a + bx + cx^2$. Then $I(p) = a + \frac{b}{2} + \frac{c}{3}$. So

$$[I][p] = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + \frac{b}{2} + \frac{c}{3}.$$